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COMMON FIXED POINT FOR WEAKLY INCREASING MAPS IN PARTIALLY ORDERED METRIC SPACES

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ABSTRACT
The objective of this paper is to establish some common fixed point theorems for weakly increasing maps in partially ordered complete metric spaces satisfying a Geraghty's type contraction. In fact here we generalized the earlier fixed point results of Gordji et al. [6].

KEYWORDS: Fixed point, common fixed point, partially ordered metric spaces, weakly increasing map.
AMS Subject classification: Primary 54H25, Secondary 47H10.

1. INTRODUCTION
In few recent years many outcomes related to fixed point, coincidence point and common fixed point for some kinds of contractions in partially ordered metric spaces have been obtained. The existence of fixed points in partially ordered sets has been initiated by Ran et al. [11] with some applications to matrix equations. After that, Nieto et al. [8, 9] extended the outcomes of Ran et al. [11]. Further, Altun & Simsek [3] introduced the concept of weakly increasing maps and obtained certain results regarding to common fixed point in ordered metric spaces.

Banach contraction principle in a complete metric space by replacing the Cauchy condition for convergence of a contractive iteration by an equivalent functional condition is achieved by Geraghty [5]. Further, Harandi et al. [4] extend the result of Geraghty [5] for generalized contraction in partially ordered complete metric space. In recent times, Gordji et al. [6] generalized the result of Harandi et al. [4] in the partially ordered complete metric spaces.

The main purpose of this paper is to obtain some generalizations of Gordji et al. [6] for common fixed point theorems for weakly increasing maps in partially ordered complete metric spaces satisfying generalized contraction.

2. PRELIMINARIES
We start this section by some basic notations, definitions and results which are used in sequel.

Definition 2.1: Let $(X, \leq)$ be a partially ordered set and $f, g: X \to X$ are said to be
(2.1.1) weakly increasing mapping if $fx \leq gfx$ and $gx \leq fgx$ for all $x \in X$. ([3])
(2.1.2) partially weakly increasing mapping if $fx \leq gfx$ for all $x \in X$. ([1])

Note that pair (f, g) is weakly increasing if and only if ordered pair (f, g) and (g, f) are partially weakly increasing but two weakly increasing maps need not be increasing.

Further, Nashine et.al [7] gave

Definition 2.2:[7] Let $(X, d, \leq)$ be a partially ordered metric spaces. Then X is regular if
(2.2.1) if a non-decreasing sequence $\{x_n\}$ in X exists such that $x_n \to x \in X$ as $n \to \infty$ then $x_n \leq x, \forall n$.

Geraghty [5] generalized the Banach contraction principle in metric spaces and proved that

If S is the family of functions $\alpha: R^+ \to [0, 1)$ such that $\alpha(t_n) \to 1$ implies $t_n \to 0$;

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Theorem 2.3:[5] Let $f: X \to X$ be a contraction of a complete metric space X satisfying
(2.3.1) $d(fx, fy) \leq \alpha(d(x,y))d(x,y), \quad \forall x, y \in X$, where $\alpha \in S$ which need not be continuous. Then for any arbitrary point x_0 the iteration $x_n = f(x_{n-1}), \; n \geq 1$ converges to a unique fixed point of f in X.

Further, Harandi et al. [4] generalized Theorem 2.3 in partially ordered metric spaces.

Theorem 2.4:[4] Let $(X, \leq)$ be a partially ordered set and there exists a metric d in X such that (X, d) is a complete metric space. Let $f: X \to X$ be a non-decreasing mapping such that there exists $x_0 \in X$ with $x_0 \leq fx_0$ satisfying (2.3.1) $\forall x, y \in X$ with $x \geq y$ and
(2.4.1) either f is continuous or (2.2.1) holds.
(2.4.2) $\forall x, y \in X$ there exists $u \in X$ which is comparable to x and y. Then f has a unique fixed point.

Recently, Gordji et al. [6] generalized the result of Harandi et.al [4] in partially ordered complete metric spaces and proved that:

Let Ψ is the class of the functions $\psi : [0, \infty) \to [0, \infty)$ such that
(i) ψ is non-decreasing; (ii) ψ is continuous; (iii) $\psi(t) = 0$ iff $t = 0$; (iv) $\psi(s + t) \leq \psi(s) + \psi(t)$.

Theorem 2.5:[6] Let $(X, \leq)$ be a partially ordered set and there exists a metric d in X such that (X, d) is a complete metric space. Let $f: X \to X$ be a non-decreasing mapping such that there exists $x_0 \in X$ with $x_0 \leq fx_0$ satisfying (2.4.1), (2.4.2) $\forall x, y \in X$ with $x \geq y$,
(2.5.1) $\psi(d(fx, fy)) \leq \alpha\Big(\psi\big(d(x,y)\big)\Big)\psi\big(d(x,y)\big)$, where $\alpha \in S, \; \psi \in \Psi$.
Then f has a unique fixed point.

Lemma 2.6:[10] Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X such that $d(x_{n+1}, x_n)$ is non-increasing and $d(x_{n+1}, x_n) = 0$
If $\{x_{2n}\}$ is not a Cauchy sequence then there exists an $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tends to ϵ as $k \to \infty$;
(2.6.1) $\{d(x_{2m_k}, x_{2n_k})\}, \{d(x_{2m_k}, x_{2n_k+1})\}, \{d(x_{2m_k-1}, x_{2n_k})\}, \{d(x_{2m_k-1}, x_{2n_k+1})\}$.

3. MAIN RESULT

We generalize Theorem 2.5, for weakly increasing maps in partially ordered complete metric space.

Theorem 3.1: Let $(X, \leq)$ be a partially ordered set and there exists a metric d in X such that (X, d) is a complete metric space. Let $f, g : X \to X$ are weakly increasing maps such that $fX \subseteq gX$ satisfying
(3.1.1) $\psi\big(d(fx, gy)\big) \leq \alpha\Big(\psi\big(M(x,y)\big)\Big)\psi\big(M(x,y)\big)$, where

$$M(x,y) = max\{\, d(x,y), d(x,fx), d(y,gy), \frac{1}{2}(d(x,gy) + d(y,fx))\}$$

for all $x, y \in X$ with $x \geq y$, $\alpha \in S$ and $\psi \in \Psi$.
(3.1.2) either f or g is continuous. Or (2.2.1) holds. Then f and g have a unique common fixed point.

Proof: Since f and g are weakly increasing mappings such that $fX \subseteq gX$ so we can construct a sequence $\{x_n\}$ in X starting with arbitrary $x_0 \in X$ such that
$x_1 = fx_0 \leq gfx_0 = gx_1, \;\; x_2 = gx_1 \leq fgx_1 = fx_2, \;\; x_3 = fx_2 \leq gfx_2 = gx_3$, inductively
$x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all $n \geq 0$. i.e. $x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots$

Now we prove that $\{x_n\}$ is a Cauchy sequence in X. For this let us consider that $d(x_{2n}, x_{2n+1}) > 0$ for every n. If not then $x_{2n} = x_{2n+1}$, for some n, therefore using (3.1.1), we have

$$\psi\big(d(x_{2n+1}, x_{2n+2})\big) = \psi\big(d(fx_{2n}, gx_{2n+1})\big) \leq \alpha\Big(\psi\big(M(x_{2n}, x_{2n+1})\big)\Big)\psi\big(M(x_{2n}, x_{2n+1})\big),$$

where

$$M(x_{2n}, x_{2n+1}) = max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, fx_{2n}), d(x_{2n+1}, gx_{2n+1}), \frac{1}{2}(d(x_{2n}, gx_{2n+1}) + d(x_{2n+1}, fx_{2n}))\}$$

$$= max\{d(x_{2n},x_{2n+1}), d(x_{2n},x_{2n+1}), d(x_{2n+1},x_{2n+2}), \frac{1}{2}(d(x_{2n},x_{2n+2}) + d(x_{2n+1},x_{2n+1}))\}$$

$$= max\{0, 0, d(x_{2n+1},x_{2n+2}), \frac{1}{2}(d(x_{2n},x_{2n+2}) + 0)\}$$
$$= d(x_{2n+1},x_{2n+2}).$$

Hence $\psi\big(d(x_{2n+1},x_{2n+2})\big) \le \alpha\Big(\psi\big(d(x_{2n+1},x_{2n+2})\big)\Big)\psi\big(d(x_{2n+1},x_{2n+2})\big)$.
Since $0 \le \alpha < 1$, we have $\psi(d(x_{2n+1},x_{2n+2})) < \psi(d(x_{2n+1},x_{2n+2}))$ a contradiction.
Hence $x_{2n+1} = x_{2n+2}$. Using the similar arguments, we obtain $x_{2n+2} = x_{2n+3}$ and so on. Thus $\{x_n\}$ turns out to be a constant sequence and x_{2n} is the common fixed point of f and g.

Now $d(x_{2n},x_{2n+1}) > 0$ for every n, since $x = x_{2n}$ and $y = x_{2n+1}$ are comparable so using (3.1.1), we have

$$\psi\big(d(x_{2n+1},x_{2n+2})\big) = \psi\big(d(fx_{2n},gx_{2n+1})\big)$$
$$\le \alpha\Big(\psi\big(M(x_{2n},x_{2n+1})\big)\Big)\psi\big(M(x_{2n},x_{2n+1})\big) \tag{3.1}$$

where

$$M(x_{2n},x_{2n+1}) = max\{d(x_{2n},x_{2n+1}), d(x_{2n},fx_{2n}), d(x_{2n+1},gx_{2n+1}), \frac{1}{2}(d(x_{2n},gx_{2n+1}) + d(x_{2n+1},fx_{2n}))\}$$

$$= max\{d(x_{2n},x_{2n+1}), d(x_{2n},x_{2n+1}), d(x_{2n+1},x_{2n+2}), \frac{1}{2}(d(x_{2n},x_{2n+2}) + d(x_{2n+1},x_{2n+1}))\}$$
$$= max\{d(x_{2n},x_{2n+1}), d(x_{2n+1},x_{2n+2})\}$$

Now
$M(x_{2n},x_{2n+1}) =$ either $d(x_{2n+1},x_{2n+2})$ or $d(x_{2n},x_{2n+1})$
If $M(x_{2n},x_{2n+1}) = d(x_{2n+1},x_{2n+2})$ then from (3.1), we have
$\psi\big(d(x_{2n+1},x_{2n+2})\big) \le \alpha\Big(\psi\big(d(x_{2n+1},x_{2n+2})\big)\Big)\psi\big(d(x_{2n+1},x_{2n+2})\big)$, since $0 \le \alpha < 1$, we have
$\psi(d(x_{2n+1},x_{2n+2})) < \psi(d(x_{2n+1},x_{2n+2}))$ which is a contradiction.
Hence $M(x_{2n},x_{2n+1}) = d(x_{2n},x_{2n+1})$ and from (3.1), we have

$$\psi\big(d(x_{2n+1},x_{2n+2})\big) \le \alpha\Big(\psi\big(d(x_{2n},x_{2n+1})\big)\Big)\psi\big(d(x_{2n},x_{2n+1})\big) \tag{3.2}$$

Since $0 \le \alpha < 1$, we have $\psi(d(x_{2n+1},x_{2n+2})) \le \psi(d(x_{2n},x_{2n+1}))$.
By similar argument for $x = x_{2n-1}$ and $y = x_{2n}$, we have
$\psi(d(x_{2n},x_{2n+1})) \le \psi(d(x_{2n-1},x_{2n}))$

Hence for any n, $\psi\big(d(x_{2n+1},x_{2n+2})\big) \le \psi\big(d(x_{2n},x_{2n+1})\big) \le \psi\big(d(x_{2n-1},x_{2n})\big)$ ... implies that the sequence $\{\psi(d(x_n,x_{n+1}))\}$ is monotonically non-increasing sequence.
Hence there exists $r \ge 0$ such that $\psi(d(x_n,x_{n+1})) = r$ (3.3)
From (3.2), we have

$$\frac{\psi(d(x_{2n+1},x_{2n+2}))}{\psi(d(x_{2n},x_{2n+1}))} \le \alpha\Big(\psi\big(d(x_{2n},x_{2n+1})\big)\Big) < 1$$

Letting $n \to \infty$ and using (3.3), we have $\alpha(\psi(d(x_{2n},x_{2n+1}))) = 1$, since $\alpha \in S$ yields that $r = 0$, consequently $\psi(d(x_n,x_{n+1})) = 0$.

Now we claim that $\{x_{2n}\}$ is a Cauchy sequence. Suppose on the contrary that $\{x_{2n}\}$ is not a Cauchy sequence and using Lemma 2.6 there exist an $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to ϵ when $k \to \infty$:

$$\{d(x_{2m_k},x_{2n_k})\}, \{d(x_{2m_k},x_{2n_k+1})\}, \{d(x_{2m_k-1},x_{2n_k})\}, \{d(x_{2m_k-1},x_{2n_k+1})\}.$$

Using (3.1.1) for $x = x_{2m_k-1}$ and $y = x_{2n_k}$, we have

$$\psi\left(d\left(x_{2m_k},x_{2n_k+1}\right)\right) = \psi\left(d\left(fx_{2m_k-1},gx_{2n_k}\right)\right)$$
$$\le \alpha\left(\psi\left(M\left(x_{2m_k-1},x_{2n_k}\right)\right)\right)\psi\left(M\left(x_{2m_k-1},x_{2n_k}\right)\right)$$

where

$$M(x_{2m_k-1},x_{2n_k}) = max\{ d(x_{2m_k-1},x_{2n_k}), d(x_{2m_k-1},fx_{2m_k-1}), d(x_{2n_k},gx_{2n_k}), \frac{1}{2}(d(x_{2m_k-1},gx_{2n_k}) + d(x_{2n_k},fx_{2m_k-1}))\}$$

$$= max\{ d(x_{2m_k-1},x_{2n_k}), d(x_{2m_k-1},x_{2m_k}), d(x_{2n_k},x_{2n_{k+1}}), \frac{1}{2}(d(x_{2m_k-1},x_{2n_{k+1}}) + d(x_{2n_k},x_{2m_k}))\}$$

Letting $k \to \infty$ and using lemma 2.6, we have

$$\lim_{n\to\infty} M(x_{2m_k-1},x_{2n_k}) = \left\{\epsilon, 0,0, \frac{(\epsilon+\epsilon)}{2}\right\} = \epsilon.$$

so $\frac{\psi(d(x_{2m_k},x_{2n_k+1}))}{\psi(M(x_{2m_k-1},x_{2n_k}))} \le \alpha(\psi(M(x_{2m_k-1},x_{2n_k}))) \le 1$ using the fact that $\epsilon = \lim_{n\to\infty} d(fx_{2m_k-1},gx_{2n_k}) = \lim_{n\to\infty} M(x_{2m_k-1},x_{2n_k})$, we get $\alpha(\psi(M(x_{2m_k-1},x_{2n_k}))) = 1$, Since $\alpha \in S$, hence $\psi(M(x_{2m_k-1},x_{2n_k})) = 0$. Since ψ is a continuous mapping, $\psi(\epsilon) = 0$ and so $\epsilon = 0$ which contradicts of $\epsilon > 0$ and shows $\{x_{2n}\}$ is a Cauchy sequence. By completeness of X there exist a point $u \in X$ such that $\{x_n\}$ and its subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are also converges to u.

Suppose that f is continuous since $x_{2n} \to u$ so we have $x_{2n+1} = fx_{2n} \to fu$. On the other hand since $x_{2n+1} \to u$ it follows that $fu = u$ now, since $u \le u$ taking $x = y = u$ in (3.1.1), we have

$$\psi(d(fu,gu)) \le \alpha\left(\psi(M(u,u))\right)\psi(M(u,u)),$$

where

$$M(u,u) = max\{ d(u,u), d(u,fu), d(u,gu), \frac{1}{2}(d(u,gu) + d(u,fu))\}$$

$$= max\{ d(u,u), d(u,u), d(u,gu), \frac{1}{2}(d(u,gu) + d(u,u))\}$$

$$= d(u,gu)$$

Hence

$$\psi(d(gu,u)) \le \alpha\left(\psi(d(u,gu))\right)\psi(d(u,gu)) < \psi(d(u,gu)),$$ yields that $gu = u$.

Again suppose that g is a continuous since $x_{2n+1} \to u$ so we obtain $x_{2n+2} = gx_{2n+1} \to gu$. On the other hand since $x_{2n+2} \to u$ it follows that $gu = u$ now, since $u \le u$ taking $x = y = u$ in (3.1.1), we have

$$\psi(d(fu,gu)) \le \alpha\left(\psi(M(u,u))\right)\psi(M(u,u)),$$

where

$$M(u,u) = max\{d(u,u), d(u,fu), d(u,gu), \frac{1}{2}(d(u,gu) + d(u,fu))\}$$

$$= max\{d(u,u), d(u,fu), d(u,u), \frac{1}{2}(d(u,u) + d(u,fu))\}$$

$$= d(u,fu)$$

Hence

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$$\psi\big(d(fu,u)\big) \le \alpha\Big(\psi\big(d(u,fu)\big)\Big)\psi\big(d(u,fu)\big) < \psi(d(u,fu)),$$ yields that $fu = u$.

Now if (2.2.1) holds, then

$$d(fu,u) \le d(fu,gx_{2n}) + d(gx_{2n},u)$$

Since ψ is non-decreasing and sub-additive so we have

$$\psi(d(fu,u)) \le \psi(d(fu,gx_{2n})) + \psi(d(gx_{2n},u))$$
$$\le \alpha(\psi(M(u,x_{2n})))\psi(M(u,x_{2n})) + \psi(d(x_{2n+1},u))$$

where

$$M(u,x_{2n}) = max\,\{d(u,x_{2n}), d(u,fu), d(x_{2n},gx_{2n}), \frac{1}{2}(d(u,gx_{2n}) + d(x_{2n},fu))\}$$

$$= max\,\{d(u,x_{2n}), d(u,u), d(x_{2n},x_{2n+1}), \frac{1}{2}(d(u,x_{2n+1}) + d(x_{2n},u))\}$$

$$= max\,\{d(u,x_{2n}), d(u,u), d(x_{2n},x_{2n+1}), \frac{1}{2}(d(x_{2n},x_{2n+1}))\}$$

$$= max\,\{d(u,x_{2n}), 0, d(x_{2n},x_{2n+1})\}$$

Since $d(u,x_{2n}) \to 0$, $d(u,x_{2n+1}) \to 0$, $d(x_{2n},x_{2n+1}) \to 0$ as $n \to \infty$

Hence

$$\psi\big(d(fu,u)\big) \le \psi(0) + \psi(0) = 0.$$

i,e $\psi\big(d(fu,u)\big) = 0$ if and only if $d(fu,u) = 0$ therefore $fu = u$. By using similar argument we have $gu = u$. Thus u is a common fixed point of f and g.

For the uniqueness suppose that u and v be any two common fixed points of f and g then from (3.1.1), we have

$$d(fu,gv) \le \alpha\big(M(u,v)\big)M(u,v)$$

where

$$M(u,v) = max\{\, d(u,v), d(u,fu), d(v,gv), \frac{1}{2}(d(u,gv) + d(v,fu))\}$$

$$= max\left\{\, d(u,v), 0,0, \tfrac{1}{2}\big(d(u,v) + d(u,v)\big)\right\} = d(u,v)$$

Hence

$d(u,v) \le \alpha\big(d(u,v)\big)d(u,v) < d(u,v)$, yields that $u = v$ i.e. common fixed points of f and g is unique

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